Vortex Induced Wing Loads

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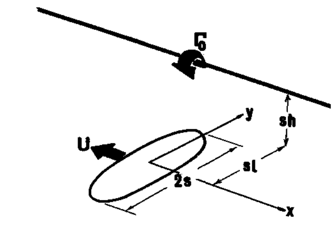

Fig. 1 Elliptic wing of aspect ratio A flying parallel to an infinite line vortex of strength Γ_o.

A LIFTING surface in uniform flight through the downwash field induced by an infinitely long vortex filament may serve to model a number of phenomena: wake penetration, wing-tail interference, formation flying, rotor aerodynamics. A solution for an important special case—flight path parallel to the vortex (Fig. 1)—was recently presented in these notes by W. P. Jones.[1] It should be remarked that for this particular case the *total* vortex induced lift and rolling moment can be obtained exactly (within the framework of linearized lifting surface theory) by use of the reciprocity relations for wings in nonuniform downwash fields.[2] The vortex induced lift corresponding to the elliptic wing case treated by Jones is, in fact, explicitly given in one of Heaslet and Spreiter's examples (Ref. 2, Eq. 59). The analogous rolling moment may also be readily deduced from their results (Ref. 2, Eq. 69).

If the spanwise distribution of the loading is required, it must be calculated directly. Jones, for example, used lifting line theory to calculate the span loading on elliptic wings in incompressible flow.[1] The rest of this Note gives these same results in an alternate simpler form more suitable for computation.

The elliptic wing of aspect ratio A and semispan s is flying parallel to a vortex of strength Γ_o displaced by lateral distance sl and vertical distance sh from midspan. It is customary to express the spanwise distribution of circulation by the Fourier series

$$\Gamma(y) = \sum_0^\infty a_n \sin(n+1)\theta \qquad (1)$$

where $\theta = \cos^{-1}(y/s)$. The Fourier coefficients a_n must be such that the wake induced downwash just cancels out the vortex induced vertical velocity

$$w(y) = (\Gamma_o/2\pi s)(l+y)/[(l+y)^2+h^2] \qquad (2)$$

at the wing.

Jones[1] gave a recursive method for calculating the a_n. Explicit expressions may however be obtained by means of the method of Ref. 3: using the vertical velocity distribution Eq. (2) in Eq. (8) of Ref. 3 these are

$$a_{2n} = (-1)^{n+1}\frac{8\Gamma_o}{\pi}\frac{r^{2n+1}\sin(2n+1)\phi}{A+2(2n+1)}$$

$$a_{2n+1} = (-1)^n\frac{8\Gamma_o}{\pi}\frac{r^{2n+2}\cos(2n+2)\phi}{A+2(2n+2)}$$

where

$$r^2 = h^2+l^2+G-(2)^{1/2}[l-(G-H)^{1/2}+h(\operatorname{sgn} H)(G+H)^{1/2}]$$

$$\tan\phi = \frac{l-[(G-H)/2]^{1/2}}{h-(\operatorname{sgn} H)[(G+H)/2]^{1/2}}$$

$$H = 1+h^2-l^2$$

$$G = [H^2+4h^2l^2]^{1/2}$$

Substitution into Eq. (1) gives the expression

$$\Gamma/\Gamma_o = -8/\pi\sum_0^\infty(-1)^n r^{2n+1}\left[\frac{\sin(2n+1)\phi\sin(2n+1)\theta}{A+2(2n+1)} - r\frac{\cos(2n+2)\phi\sin(2n+2)\theta}{A+(4(n+1)}\right] \qquad (3)$$

for the vortex induced circulation. Values computed using Eq. (3) agree with Figs. 2 and 3 of Ref. 1.

The corresponding vortex induced increments in wing lift and rolling moment coefficients may be expressed in the form

$$\Delta C_L = C_{L_\alpha}\Gamma_o(\pi s U)^{-1}\{l-[(G-H)/2]^{1/2}\} \qquad (4)$$

$$\Delta C_l = 4C_{l_p}\Gamma_o(\pi s U)^{-1}\{1+2h^2+l[2(G-H)]^{1/2} - h(\operatorname{sgn} H)[2(G+H)]^{1/2}\} \qquad (5)$$

where

$$C_{L_\alpha} = 2\pi A/(A+2)$$

is the wing lift-curve slope in uniform flow, and

$$C_{l_p} = -(\pi/4)A/(A+4)$$

is the wing roll-damping derivative.

It may be verified that Eqs. (4) and (5) are identical to expressions calculated using the reciprocity relations of Ref. 2. Using appropriate values for C_{L_α} and C_{l_p} the vortex induced contributions to lift and rolling moment may be estimated from Eqs. (4) and (5) for wings of nonelliptical planform.

Finally, it should be noted that some account of the viscous core in a real vortex may be introduced in these results through the artifice of using "effective" values for the vortex location and circulation.[4,5]

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Computation of Transonic Flow about Finite Lifting Wings

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RELAXATION methods have proven to be an accurate and effective means for computing the transonic flowfield around two-dimensional airfoils when the freestream is subsonic. Murman and Cole[1] have developed such a technique for solving

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the transonic small disturbance equation for the velocity potential for two-dimensional nonlifting airfoils, and Murman and Krupp[2,3] have extended the method to the lifting case. Even for blunt airfoils, these results compare favorably with relaxation solutions of the full equations obtained by Garabedian and Korn,[4] Steger and Lomax,[5] and Jameson.[6] Bailey and Steger[7] have recently presented some results for three-dimensional lifting wings. They employed a hybrid system based on the small disturbance equation written in terms of the velocity potential for part of the computation and in terms of the velocity components for the remainder. This Note presents some comments and preliminary results obtained by a relaxation solution of the small disturbance equation for the velocity potential for three-dimensional wings. The method is a natural extension of the two-dimensional work of Refs. 1–3 and, consequently, differs in many respects from the method used in Ref. 7.

Description of Method

The three-dimensional equation for the nondimensional perturbation velocity potential $\phi(x, y, z)$ can be written as

$$[\beta^2-(\gamma+1)M_\infty{}^2\phi_x]\phi_{xx}+\phi_{yy}+\phi_{zz}=0 \qquad (1)$$

where $\beta^2 = 1-M_\infty{}^2$, M_∞ is the freestream Mach number, γ is the ratio of specific heats, and subscripts denote partial differentiation. The flow tangency condition on the wing is

$$\phi_z(x,y,0\pm)=(1+\phi_x)F_x(x,y)-\alpha,\ x,y \text{ on wing planform} \qquad (2)$$

where $z = F(x, y)$ is the equation of the wing surface and α is the angle of attack. Equation (2) has been used for blunt leading-edge wings, whereas the ϕ_x term has been omitted for sharp leading-edge wings. In either case, the boundary condition (2) is applied in a plane which is located midway between the two central grid planes in the z direction as in Ref. 2.

For a lifting wing, the trailing vortex sheet is assumed to lie in the plane of the wing from the trailing edge to infinity. The boundary condition there requires that the jump in potential across the vortex sheet is independent of x; thus,

$$\phi(x,y,0+)-\phi(x,y,0-)=\Gamma(y),\ x,y \text{ on vortex sheet} \qquad (3)$$

where $\Gamma(y)$, the circulation at section y, is determined as part of the solution.

The disturbance velocity potential is required to vanish at infinity. The computational grid can be reduced in extent by making use of an analytical representation of the far field to satisfy, in effect, this boundary condition. This procedure was first used by Murman and Cole[1] and in subsequent small-disturbance computations of Refs. 2 and 3. Klunker[8] has derived analytical expressions for the potential far from thin lifting wings at transonic speeds. In the present work, only the term due to lift (which dominates) is used; the potential $\phi(x, y, z)$ is matched to the analytical expressions for ϕ_{lift} at the outer grid boundaries (located at large but finite values of x, y, z). For a nonlifting wing, $\phi(x, y, z)$ is set to zero at the outer grid boundaries.

An analytical coordinate transformaton of the form $\xi = \xi(x)$, $\eta = \eta(y)$, $\zeta = \zeta(z)$ has been used to obtain a relatively dense grid on the wing and a much coarser grid far from the wing. The computational coordinates ξ, η, ζ are evenly spaced, though not equally so in all directions. Use of a coordinate transformation is not entirely satisfactory since a) it is difficlt to locate the grid points where desired, and b) too rapid a change in the derivatives of the transformation can lead to anomalies in the solution.

Finite differences are written in the ξ, η, ζ space. For derivatives in the η and ζ direction, second-order accurate three-point central differences are used. For derivatives in the streamwise (ξ) direction, Murman and Cole[1] found that a mixed difference scheme is the key to success for transonic flows; a modification similar to that used in Refs. 4 and 6 has been used here. Central differences are used at all grid points for ϕ_ξ and at subsonic points for $\phi_{\xi\xi}$; at supersonic points a backward difference for $\phi_{\xi\xi}$ is used which is a weighted average of three- and four-point difference formulas. Use of central differences everywhere for ϕ_ξ linearizes the difference equations, so that inner iterations are unnecessary.

The circulation at each wing section, $\Gamma(y)$, must be determined as part of the iterative procedure in the velocity potential formulation. In contrast, the Kutta condition can be treated much like the surface boundary condition in the velocity component formulation of Ref. 7. Nevertheless, the basic simplicity and reduced computer storage requirements favor the use of the velocity potential. In the present computations, the circulation has been relaxed along with the velocity potential.

Line relaxation (along lines ξ and η constant) is used to solve the difference equations. Overrelaxation is used for each line unless it contains a supersonic point, in which case it is underrelaxed. The exterior boundary planes are updated with the far-field solution using current values of $\Gamma(\eta)$. The jump in potential at the trailing edge of each spanwise station is obtained by extrapolation and is used with the old value of $\Gamma(\eta)$ in a standard relaxation form to obtain a new value of $\Gamma(\eta)$.

Results

The first computations made using the present method were for a wing with a symmetric biconvex airfoil section in a subsonic flow at zero incidence. The results showed considerable fore-aft asymmetry (even with a symmetric computational network), part of which could be related to a bias built into the computations by sweeping through the computational mesh in the streamwise direction for all cycles. This asymmetry in C_p was reduced by nearly an order of magnitude, when the ϕ correction ($\Delta\phi$ between iteration cycles) was about 10^{-4}, by sweeping through the computational mesh first downstream and then upstream on alternate iteration cycles. When the ϕ correction reaches about 10^{-6}, there is but a moderate improvement in the symmetry of the solution obtained by using the alternate sweep procedure. The alternate sweep has been used successfully for supercritical flows with shock waves; however, the convergence rate for different cases has shown both increases and decreases compared with that

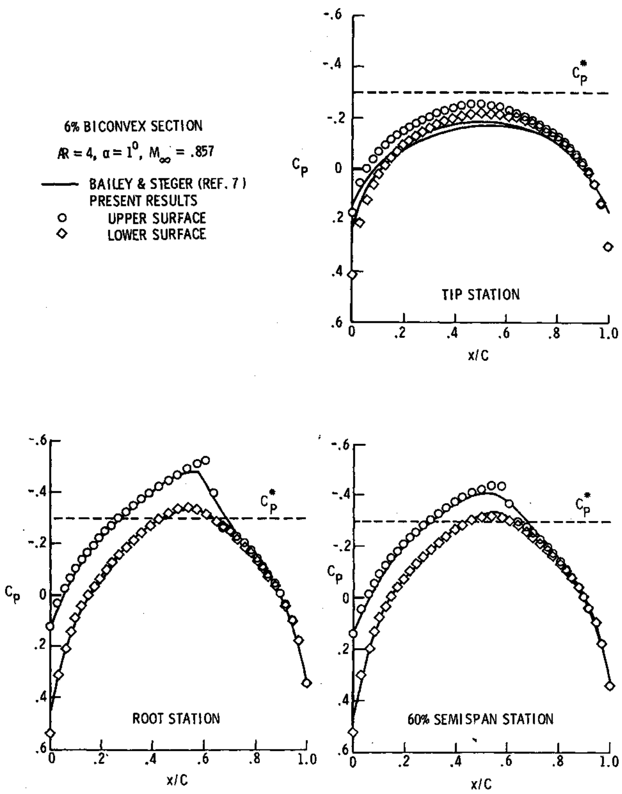

Fig. 1 Computed surface pressure coefficients for supercritical lifting rectangular wing with sharp leading edge.

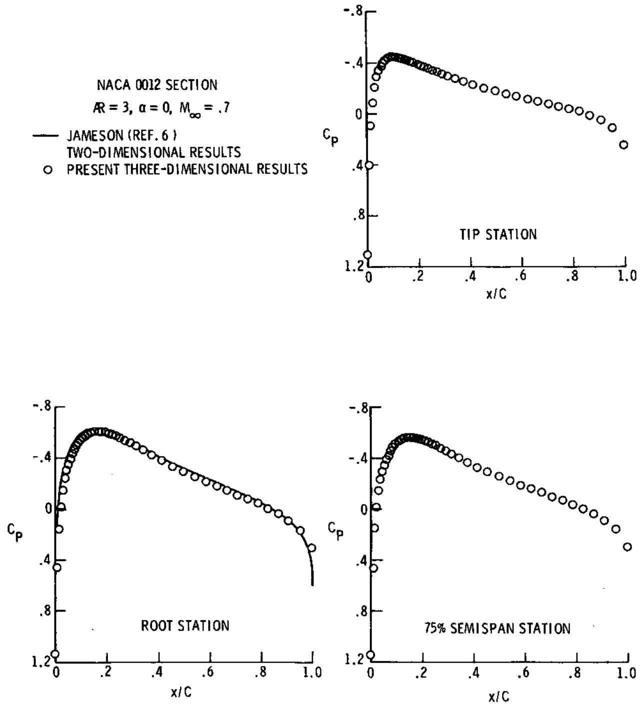

Fig. 2 Computed surface pressure coefficients for subcritical nonlifting rectangular wing with blunt leading edge.

required for similar computations with streamwise sweep only.

A comparison is made in Fig. 1 between the surface pressure coefficients obtained using the present method and those of Bailey and Steger.[7] These results are for a rectangular wing of aspect ratio 4 with a 6% thick biconvex airfoil section at 1° angle of attack and a freestream Mach number of 0.857. The value of critical pressure shown (C_p*) corresponds to the condition where the coefficient of ϕ_{xx} in Eq. (1) vanishes and, consequently, differs from the exact value. Agreement at the root and 60% spanwise station is very good; the differences in the region of the shock on the upper surface can be attributed to a greater resolution of the present results. Larger differences occur at the tip station and it should be noted that the loading does not vanish there for either solution. In the present computations, this tip station was treated the same as the other wing stations: that is, a zero load condition was not imposed at the tip. A nonzero load there implies that the wing tip is, in effect, between grid points; thus there is an uncertainty regarding the wing span. At the present stage of development, it is not clear how the tip region should be treated or how important the tip singularity is to the global flowfield solution. The computational mesh used for the present calculations shown in Fig. 1 contained $60 \times 22 \times 40$ grid points in the x, y, and z directions, respectively. There were 37 grid points on the airfoil at each spanwise station with 11 stations on the wing semispan.

Results of a subcritical ($M_\infty = 0.7$) three-dimensional calculation for a wing with a blunt leading edge at zero lift are shown in Fig. 2. The wing planform is again rectangular with an aspect ratio of 3, and the airfoil section is the NACA 0012. The pressure distribution at the wing root is compared with the very refined two-dimensional calculations of Jameson[6]; the agreement is quite good, although the very rapid expansion at the leading edge is not fully obtained in the three-dimensional computation. This is due in part to an insufficient number of grid points very near the leading edge. Similar computations for the same three-dimensional wing at lift and in supercritical flows showed much larger discrepancies in the leading-edge region using the same computational mesh. A successful computation for a blunt airfoil with the small disturbance equations requires that several grid points be located in the leading-edge region where the slope is large, and lack of such resolution causes errors in the rapid expansion in the leading-edge region and gives only a qualitatively correct behavior elsewhere.

The computations shown in Fig. 2 used a grid of $85 \times 20 \times 40$ in the x, y, and z directions, respectively, and required 300 K octal machine storage. Forty-eight of the 85 were located on the wing at each of 14 spanwise stations. A total of 85 grid points in the streamwise direction appears adequate for blunt airfoil computations; however, a larger percentage of them should be located on the wing with a higher density in the leading-edge region. Location of individual grid points to achieve a high density in the leading-edge region without excessively large higher derivatives in the coordinate transformation has proved difficult. However, these difficulties are computational in nature and are not inherent to the relaxation techniques or to the small-disturbance equations. It appears entirely feasible to calculate supercritical flows about wings of simple shape with blunt airfoil sections on present generation computers.

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Eigenvalues and Eigenvectors for Solutions to the Radiative Transport Equation

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Introduction

THE transport equation for isotropic axisymmetric radiative transfer has the form

$$\frac{dI}{d\tau}(\tau,\mu) = \frac{-I(\tau,\mu)}{\mu} + \frac{W}{2\mu}\int_{-1}^{1} I(\tau,\mu')\,d\mu' + \frac{(1-W)}{\mu}n^2 I_b(T) \tag{1}$$

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